Eilenberg-Moore Bicategories for Op-monoidal Pseudomonads

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Main result: Let $(\mathcal{A}, \otimes, I)$ be a (symmetric) monoidal bicategory, (T, μ, η) an opmonoidal pseudomonad on it. Then the (symmetric) monoidal structure of E lifts to a (symmetric) monoidal structure on the bicategory of Eilenberg-Moore pseudoalgebras $\mathcal{A}^T$.*

*And dual compatibility structure allows (resp. symmetric) monoidal structure to extend to Kleisli bicategories.

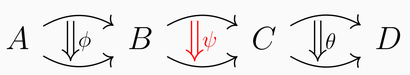
Steps in the 1D setting

1. $(\mathbf{Mnd}(\mathbf{Cat}), \times, \mathbf{1})$ is a monoidal 2-category.
2. $\mathbf{Pseudomonoids}(\mathbf{Mnd}(\mathbf{Cat}), \times, \mathbf{1}) \cong \mathbf{Mnd}(\mathbf{MonCat}_{oplax})$.
3. $\mathbf{EM} : \mathbf{Mnd}(\mathbf{Cat}) \to \mathbf{Cat}^{\mathbf{2}}$ is normal lax monoidal.
4. Hence monoidal category structures lift to algebras for monads.
5. Analogously, braid ($\beta_{X,Y} : X \otimes Y \to Y \otimes X$) and symmetry ($\beta_{X,Y} = \beta^{-1}_{Y,X}$) structures also lift since $\mathbf{EM}$ is a symmetric monoidal 2-functor.

The challenge is to replicate these results in the two-dimensional setting without getting bogged down with 3D symmetric monoidal categories.

Step 1. The Gray tensor product of pseudomonads

- Gambino and Lobbia: $\mathbf{EM} : \mathbf{Psmnd}(\mathbf{Gray}) \to \mathbf{Gray}^{\mathbf{2}}$ is tri-fully faithful.
- Restrict to the 2-category $\mathbf{Psmnd}(\mathbf{Gray})_{ast}$ of 2-cells ψ which satisfy middle-four interchange on either side. That is, the interchangers θ_ψ and ψ_ϕ are both identities.



- **Theorem:** This is a (symmetric) monoidal 2-category when equipped with the **Gray**-tensor product of pseudomonads.

Step 2. The tricategory of op-monoidal pseudomonads, and a strictification result

- Garner and Gurski: $\mathbf{Tricat}_{oplax}$ is a (large) tricategory. Restrict this to $\mathbf{MonBicat}_{oplax}$.
- **Definition:** An *op-monoidal pseudomonad* is a pseudomonad internal to this tricategory $\mathbf{MonBicat}_{oplax}$.
- **Theorem:** Any op-monoidal pseudomonad is monoidally biequivalent to one whose underlying monoidal bicategory is a **Gray**-monoid, and whose underlying pseudofunctor is strict.
- **Proposition:** Such structure precisely corresponds to a pseudomonoid in $(\mathbf{Psmnd}(\mathbf{Gray})_{ast}, \otimes, \mathbf{1})$.

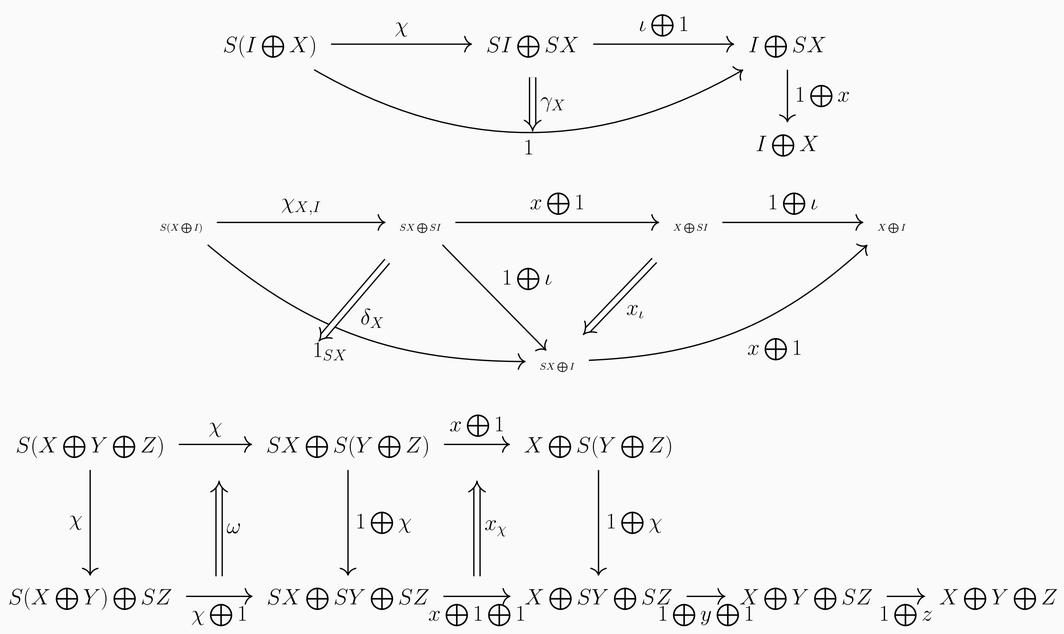
Step 3. Normal lax monoidal 2-functoriality of $\mathbf{EM} : \mathbf{Psmnd}(\mathbf{Gray})_{ast} \to \mathbf{Gray}^{\mathbf{2}}_{2-nat}$

- Restricting **EM** to these 2-cells gives 2-natural transformations.
- We must first close 2-natural transformations under **Gray**-tensor product to get a target monoidal 2-category for **EM**.
- **Theorem:** **EM** then becomes a strictly associative and unital, normal lax monoidal 2-functor.

$$\big(x : SX \to X, y : TY \to Y\big) \longmapsto (SX, TY) \xrightarrow{(x,TY)} (X, TY) \xrightarrow{(X,y)} (X, Y)$$

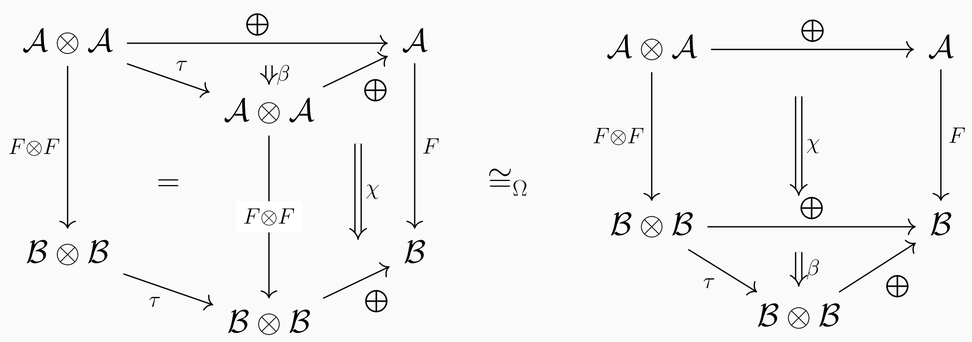
Step 4. The monoidal bicategory of pseudoalgebras

- All 2-cells in $\mathbf{Gray}^{\mathbf{2}}_{2-nat}$ involved in step 3 are in fact just 2-natural transformations, rather than **Gray**-tensor products of such.
- With some further calculation, **Theorem:** the 2-category of pseudoalgebras for an op-monoidal pseudomonad inherits a monoidal bicategory structure.
- The associator and unitors of this structure are 2-natural transformations which satisfy the usual monoidal category axioms on the nose, rather than up to isomorphism.



Step 5. Lifting symmetries

- There are tricategories of braided, sylleptic and symmetric monoidal bicategories and analogous coherence theorems to the one from Step 1.
- This allows one to define braided, sylleptic and symmetric pseudomonads
- **Proposition:** These induce data in $\mathbf{Psmnd}(\mathbf{Gray})$.
- These data should comprise braided (resp. sylleptic, symmetric) pseudomonoids in a conjectural monoidal **Gray**-category $\mathbf{Psmnd}(\mathbf{Gray})$.



- We establish enough aspects of the (conjectural) symmetric monoidal **Gray**-functoriality of $\mathbf{EM} : \mathbf{Psmnd}(\mathbf{Gray}) \to \mathbf{Gray}^{\mathbf{2}}$ to lift these data to pseudonatural equivalences and invertible modifications between 2-categories of pseudoalgebras.
- The symmetric monoidal bicategory axioms are then easy to check using faithfulness on 2-cells of the forgetful 2-functors $\mathcal{A}^S \to \mathcal{A}$.

Extending Structures to Kleisli Bicategories

- The dual case involves invertible icons which *are* closed under **Gray**-tensor products, unlike 2-natural transformations.
- However, dualising only gives an *oplax* monoidal 2-functor $\mathbf{KL} : \mathbf{Psmnd}(\mathbf{Gray})_{ast} \to \mathbf{Gray}^{\mathbf{2}}$.
- **Theorem:** A left pseudoadjoint is biessentially surjective on objects if and only if the 2-functor induced from the **Gray**-enriched Kleisli object is a biequivalence.
- Strong monoidal 2-functoriality of $\mathbf{KL} : \mathbf{Psmnd}(\mathbf{Gray})_{ast} \to \mathbf{Gray}^{\mathbf{2}}$ is then easy to check.
- Braidings, syllapses and symmetries similarly extend to **Gray**-enriched Kleisli objects for pseudomonads.
- Finally, all of these structures transport from **Gray**-enriched Kleisli objects for pseudomonads, to their Kleisli bicategories and 2-categories of free pseudoalgebras.

Potential Applications/Future Work

- Two-dimensional models of linear and differential logic.
- Three-dimensional models of TQFT.

See the preprint for references: Adrian Miranda- *Eilenberg-Moore Bicategories for Op-Monoidal Pseudomonads* `https://arxiv.org/abs/2402.11703`

This work is supported by EPSRC under grant EP/V002325/2.

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